Short Papers

One-Root Intervals of Dispersion Relations for a Resonator with Dielectric Sample

VLADIMIR BILIK

Abstract —A method is developed yielding intervals with analytically expressed bounds that contain only one root of the dispersion relation corresponding to any chosen mode of a cylindrical resonator with a dielectric sample, filling its cross section along part of the length of the resonator. A method for finding both the eigenfrequency and the sample permittivity is given.

I. Introduction

A frequently used arrangement to measure dielectric properties of materials is a resonator with a dielectric sample filling part of the resonator length in its whole cross section [1]. The general configuration is shown in Fig. 1. Such a system can support TE_{mnp} and TM_{mnp} modes. To analyze them, it is necessary to solve the following set of equations:

$$k_0^2 = k_c^2 + h_a^2 \tag{1}$$

$$\epsilon_r k_0^2 = k_c^2 + h_d^2 \tag{2}$$

$$F(h_a, h_d, \epsilon_r) = 0. \tag{3}$$

Equation (3) is generally called the dispersion relation. $k_0 = 2\pi f_0/c$, where f_0 is the eigenfrequency of a given mode, c is the velocity of light in free space, k_c is the cutoff wavenumber, h_a and h_d are the wavenumbers along the resonator's axis in air and dielectric, respectively, and ϵ_r is the relative permittivity of the sample. It is assumed that the dielectric is lossless, the resonator walls are perfectly conductive, and that the sample thickness as well as the distances $d_1 + d_2$ are nonzero. Forms of (3) which are convenient for numerical computations are given in Appendix 1. A method for deriving them has been described by Gardiol [2].

Because h_d is always real, it is convenient to eliminate all unknowns but h_d when solving (1)–(3), so that we obtain an equation with one real variable

$$G(h_d) = 0 \tag{4}$$

which is a rather complicated transcendental equation (it is sufficient to solve it for $h_d \geqslant 0$). The roots of (4) can be arranged into an increasing sequence $\{x_s\}_{s=1}^{\infty}$. The sequence number s of the root is $s = p$ for TE_{mnp} modes ($p = 1,2,3,\cdots$) and $s = p+1$ for TM_{mnp} modes ($p = 0,1,2,\cdots$). In order to numerically evaluate any of these roots, an initial guess of the desired root must be made. This guess should be sufficiently close to ensure proper convergence of the root-finding routine. It is therefore useful if one can state an interval $\langle u_s; v_s\rangle$ which contains only the required root x_s. In this paper, a method of finding such a one-root interval with analytically expressed bounds is presented. It is based on a similar method [3] developed for dielectrically loaded rectangular waveguides.

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The author is with the Electrotechnical Institute of Slovak Technical University, Vazovova 5, 812 19 Bratislava, Czechoslovakia.

II. Determination of the One-Root Interval

Equation (3) can be rewritten as

$$h_a \frac{\left[\epsilon_r + h_a^{-2} k_c^2 (\epsilon_r - 1)\right] \tan h_a d_1 \tan h_a d_2 - 1}{\tan h_a d_1 + \tan h_a d_2} = h_d / \tan h_d t \tag{5}$$

for TE modes and as

$$h_a \epsilon_r \frac{\tan h_a d_1 \tan h_a d_2 - h_a^{-2} k_c^2 \epsilon_r^{-2} (\epsilon_r - 1) - \epsilon_r^{-1}}{\tan h_a d_1 + \tan h_a d_2} = h_d / \tan h_d t \tag{6}$$

for TM modes. The right-hand side is a function of h_d and has poles at $b_j = j\pi/t$, $j = 1,2,3,\cdots$. It is decreasing within $(0; b_1)$ and $(b_j; b_{j+1})$. The left-hand sides of these equations are functions of h_a. For $h_a^2 \geqslant 0$, the left-hand side of (5) has poles at $a_i = i\pi/(d_1 + d_2)$ and that of (6) poles at $a_i = \pi(i-1)/(d_1+d_2)$, $i = 1,2,3,\cdots$. Both functions are increasing within $(a_i; a_i + 1)$. For $h_a^2 < 0$, the left-hand sides are continuously decreasing functions of $|h_a|$.

When evaluating f_0, we multiply (1) by ϵ_r, subtract it from (2), and express h_a in terms of the resulting equation. Substituting h_a into (5), resp. (6) yields the equation

$$f(h_d) = h_d / \tan h_d t. \tag{7}$$

The function $f(h_d)$ has poles at

$$c_i = \sqrt{\epsilon_r a_i^2 + k_c^2(\epsilon_r - 1)}, \qquad i = 1,2,3,\cdots$$

and is increasing within $(0; c_1)$ and $(c_i; c_{i+1})$. The plot of f and $g = h_d/\tan h_d t$ is given in Fig. 2. It is obvious that just one root of (7) thus of (4) lies between each pair of adjacent poles from the set $\{c_i, b_j\}_{i,j=1}^{\infty}$. The first root always lies between $h_d = 0$ and the lowest of the poles. Consequently, if forming a nondecreasing sequence $\{z_k\}_{k=1}^{\infty}$ from zero and *all* the elements of the sets $\{c_i\}$ and $\{b_j\}$, the root x_s will satisfy the condition $z_s \leqslant x_s \leqslant z_{s+1}$ and the bounds of the required interval are

$$u_s = z_s, \quad v_s = z_{s+1}. \tag{8}$$

Note that if $z_s = z_{s+1}$, we should choose $v_{s-1} = z_s - \delta$, $u_{s+1} = z_s + \delta$, where δ is a sufficiently small correction, because z_s is equal to the root x_s. If x_s is chosen as an initial value when looking for x_{s-1} or x_{s+1}, the iteration may fail. However, the case $z_s = z_{s+1}$ is extremely unlikely for $\epsilon_r > 1$ and can safely be disregarded. It can occur for $\epsilon_r = 1$, but this case (empty resonator) is of interest only when debugging the program.

When solving for ϵ_r (task of measuring the permittivity of the sample), we express h_a in terms of (1) and ϵ_r in terms of (2). Substituting them into (5), resp. (6), we again obtain an equation similar to (7). This time, its left-hand side f has no poles and is monotonous in intervals large enough that only one intersection of f and g occurs in each section of g. Therefore

$$j\pi/t \leqslant x_s \leqslant (j+1)\pi/t. \tag{9}$$

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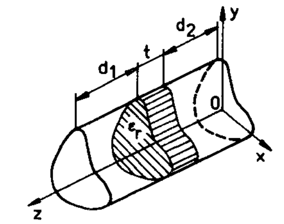

Fig 1 Configuration discussed in this paper.

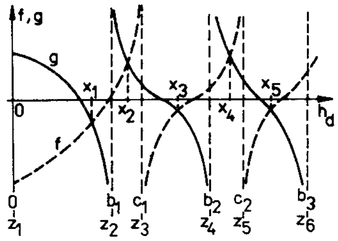

Fig. 2. Graphical solution of (7). The elements c_i, b_j, z_k are shown in the lower part of the diagram.

It can be proved (Appendix II) that for j we must take

$$j = s - i - 1 \tag{10}$$

where s is the sequence number of the sought root and

$$i = \begin{cases} \mathrm{int}[h_a(d_1 + d_2)/\pi] & h_a^2 \geqslant 0, \quad \text{TE modes} \quad (11a) \\ 1 + \mathrm{int}[h_a(d_1 + d_2)/\pi] & h_a^2 \geqslant 0, \quad \text{TM modes} \quad (11b) \\ 0 & h_a^2 < 0. \quad (11c) \end{cases}$$

(int x is the greatest integer value of x). The bounds of the required interval are

$$u_s = j\pi/t, \quad v_s = (j+1)\pi/t. \tag{12}$$

III. Conclusion

The presented method of defining one-root intervals quickly and reliably determines the roots of dispersion relations for any chosen mode of resonance without ambiguity and represents an effective tool for systematic computer analysis of the structure.

Appendix I
Dispersion Relations

In order for the transverse field components to be continuous at the air–dielectric interfaces of the resonator, certain specific relation between the wavenumbers h_a and h_d must be satisfied. This relation is referred to as a dispersion relation. For the TE modes of the discussed resonator, the dispersion relation is

$$\frac{\sin h_d t}{h_d}\left[\left(\frac{h_d}{h_a}\right)^2 \sin h_a d_1 \sin h_a d_2 - \cos h_a d_1 \cos h_a d_2\right] - \cos h_d t \frac{\sin h_a(d_1 + d_2)}{h_a} = 0 \tag{A1}$$

and for the TM modes, it is

$$\frac{\sin h_d t}{h_d}\left(h_d^2 \cos h_a d_1 \cos h_a d_2 - \epsilon_r^2 h_a^2 \sin h_a d_1 \sin h_a d_2\right) + \epsilon_r h_a \cos h_d t \sin h_a(d_1 + d_2) = 0. \tag{A2}$$

Appendix II
Proof of Formulas (10) and (11)

The lower bound $u_s = z_s$ of a one-root interval for root x_s is identical with the sth member of the sequence $\{z_k\}$, whose members z_1 through z_s can be obtained by arranging the elements

$$0, c_1, c_2, \cdots, c_i, b_1, b_2, \cdots, b_j \tag{A3}$$

into a nondecreasing (finite) sequence. The overall number of elements used in (A3) is $s = i + j + 1$, hence (10) is valid. Last used poles being b_j and c_i implies both (9) and $c_i \leqslant x_s < c_{i+1}$, which is for $h_a \geqslant a_1$ equivalent with $a_i \leqslant h_a < a_{i+1}$ and results in (11a,b) after simple calculations. If $h_a < a_1$ ($a_1 \neq 0$ for TE modes only) or if $h_a^2 < 0$, no element from the set $\{c_i\}$ is used in (A3), so that $i = 0$. The result $i = 0$ is obtained from (11a) for $h_a < a_1$ and from (11c) for $h_a^2 < 0$.

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Normal-Mode Parameters of Microstrip Coupled Lines of Unequal Width

S. KAL, D. BHATTACHARYA, AND N. B. CHAKRABORTI

Abstract—**Empirical relations for the capacitive and inductive coupling coefficients have been used to compute normal-mode parameters of nonidentical microstrip coupled lines. The computed values of mode velocities and mode impedances are compared with available results from the literature and experimental results on nonidentical microstrip couplers to test the applicability of the empirical relations.**

I. Introduction

The normal-mode parameters of microstrip coupled lines are usually determined from the capacitances and inductances of the microstrip lines [1]–[6]. These are found by solving Laplace's equation for the quasi-TEM case and the Helmholtz equation for the dispersive case [1]. Tripathi and Chang [2] calculated self and mutual capacitances for nonidentical microstrip coupled lines using Green's function integral equation method. This short paper aims at providing simple relations for finding the normal-mode parameters of nonidentical lines.

A previous communication [7] described the use of the empirical relations for inductive and capacitive coupling coefficients of identical coupled microstrip lines. These have now been modified to enable calculation of the normal-mode parameters of nonidentical microstrip coupled lines from a knowledge of the dimensional ratio of the lines and relative dielectric constant of the substrate material. Values of normal-mode parameters thus computed from the quasi-static approach have been compared with

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The authors are with the Department of Electronics and Electrical Communication Engineering, Indian Institute of Technology, Kharagpur—721302, India

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